

Analyzing Singularities of Curve Offsets: A Synergistic Approach using CAS, DGS, and Generative AI

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Abstract. *We explore envelopes of families of circles of constant radius centered on a plane curve, and offsets of plane curves, using automated commands for envelopes and geometric loci. The topology of envelopes and offsets is generally more complicated than that of the progenitor, and it is important to study the singularities. These can be cusps and crunodes (self-intersections). Work involves both analytic and algebraic methods (such as elimination), networking between different kinds of software, and automated commands to determine envelopes and geometric loci. Accurate plotting is a central issue, as in a neighborhood of a singular point the software may leave a blank gap. To address this issue, we demonstrate how the new Plot2D command provided by GeoGebra Discovery is effective. Finally, we analyze the output of generative AI for the above questions and discuss briefly how the pitfalls can and should be used to develop critical thinking and other skills. Generative AI can nevertheless be useful for rapid paraphrasing, suggesting candidate parametrizations, and proposing computational steps, provided that all results are verified by theorem-based software and mathematical reasoning. The activities presented here have been proposed to undergraduates, both in regular classes and as individual students acting as undergraduate researchers, and to in-service teachers working toward an advanced degree.*

1 Introduction

Envelopes of parameterized families of plane curves and offsets of plane curves are important domains of study, both for the mathematics involved and for applications [36]. While offsets are sometimes considered as a special case of envelopes, they are not identical (see Section 2 for precise definitions and the distinction between them). Offsets, called parallel curves by Leibniz, are geometric loci of points obtained after a nontrivial geometric construction. They may sometimes look similar to the original curve, called the *progenitor*, but this is generally not the case. The authors asked different AI conversational agents and obtained answers describing offsets of an ellipse as ellipses, which is incorrect [24]: what appears are spiric curves (also called Cassini ovals), whose implicit equations are irreducible polynomials of degree higher than 2. As the examples in Section 6 illustrate, current large language models generate plausible text from

heterogeneous training sources and do not natively perform symbolic verification; therefore, their mathematical claims require external checking.

These topics can be studied in a technology-rich environment and automated methods are developed and implemented in software. Our exploration involves a fruitful dialogue between a Dynamic Geometry System, used mostly for interactive exploration and conjectures (but not only), and a Computer Algebra System for algebraic manipulations and animations. The software that we use for Dynamic Geometry, GeoGebra¹, has a companion called GeoGebra Discovery². Actually GeoGebra has an embedded CAS called Giac [30]. Note that already in standard GeoGebra, there is a distinction between numerical and symbolic commands. For example, **Locus** is numerical and **LocusEquation** is symbolic. This is often transparent to the user. GeoGebra Discovery contains symbolic versions of GeoGebra commands, and more advanced automated commands [1, 28].

We use the commands for the computation of envelopes and geometric loci, as well as the plotting commands. If more computational power is requested, e.g. when polynomials of higher degree are involved, we use Maple. Another motivation is the limited space available in the CAS window, which makes the equations hard to read. Even if a (manual) cut-and-paste is possible, it may be not user friendly.

When studying the topology of envelopes and offsets, as in [38, 23], one of the central topics is the exploration of singular points. Different issues appear:

- (i) If the curve is given by a parametric representation, the singular points are determined by the vanishing of first derivatives. Then the nature of the singularity (cusps, crunodes, etc.) has to be determined by computing successive higher-order derivatives and checking linear independence.
- (ii) If the curve is given by a symbolic implicit equation, the singular points are determined by the vanishing of first (partial) derivatives. But now, the nature of the singularities is harder to determine.
- (iii) Because of the differences between what can be afforded from implicit equations and from parametric equations, we use Maple's commands to switch between both kinds of representation, namely **parametrization** and **implicitize**; both belong to the *algcures* package, and the second requires an initial guess of the degree of the desired polynomial.
- (iv) Plotting the curve may reveal problems: the plot is often inaccurate in a neighborhood of a singular point and a blank area may be left. Here the **Plot2D** command offered by GeoGebra Discovery is an efficient tool to fill the gaps. Maple's **plot_real_curve** command from the *algcures* package, already mentioned in item (iii), also provides accurate plotting of implicit algebraic curves and can complement **Plot2D** in a CAS-based workflow.

In this paper, we exemplify the issues mentioned above. Such examples have been proposed to in-service teachers working toward an advanced degree. The participants had almost no literacy with mathematical software, and the course was a strong “game changer” for them. They reacted with enthusiasm to the discovery that mathematics in a technology-rich environment is

¹Freely downloadable from <https://www.geogebra.org>.

²Freely downloadable from <https://github.com/kovzol/geogebra-discovery>.

not a static field of knowledge as they were taught years ago, but a dynamic and living field of knowledge construction. Three undergraduate students at the Jerusalem College of Technology (JCT) act as research assistants for the first author; their contribution yields analyses of the processes with software.

This paper makes the following specific contributions:

- (a) We demonstrate, on two non-trivial examples (the ellipse and the deltoid as progenitors), that GeoGebra Discovery's **Plot2D** command resolves known plotting failures near singular points of envelopes and offsets, filling gaps that the standard **Plot** command leaves open.
- (b) We provide a systematic, example-by-example documentation of the failure modes of several generative AI systems (ChatGPT GPT-5.2, Claude Sonnet, Grok, and Gemini) when prompted to compute envelopes and offsets of the same curves, with mathematical commentary on each error.
- (c) We describe a verification workflow in which CAS and DGS serve as ground truth against which AI outputs are checked, and show how this workflow can be embedded in mathematics education to develop critical thinking.

The paper is organized as follows. Section 2 introduces the mathematical definitions and the distinction between envelopes and offsets. Section 3 presents the methodological and cognitive framework, discussing registers of representation, human-machine communication, and the role of automated commands. Sections 4 and 5 present the computational examples for the ellipse and the deltoid respectively. Section 6 reports and analyzes our experiments with generative AI systems on the same geometric questions. Section 7 reports qualitative observations from deploying these activities with in-service mathematics teachers and undergraduate research assistants. Section 8 presents the conclusions.

2 Mathematical Background

In this section, we introduce the main mathematical objects studied in this paper: envelopes of 1-parameter families of plane curves, and offsets. We also clarify the important distinction between them, and describe the types of singularities that arise.

2.1 Envelopes

Bruce and Giblin [10, Chap. 5] give 4 non-equivalent definitions of an envelope of a 1-parameter family of plane curves. Here we will refer to two of them, with the names given by Kock [27].

Definition 1 (impredicative) *Let $\mathcal{C}_t$ be a 1-parameter family of plane curves. An envelope of this family, if it exists, is a curve tangent to all the curves of the family.*

Definition 2 (analytic) *Let $\mathcal{C}_t$ be a 1-parameter family of curves given by an equation $F(x, y, t) = 0$, where t is a real parameter. An envelope of the family $\mathcal{C}_t$ is the set of all points (x, y) in the plane verifying the system of equations*

$$\begin{cases} F(x, y, t) = 0 \\ \frac{\partial}{\partial t} F(x, y, t) = 0 \end{cases} \quad (1)$$

2.2 Offsets

Definition 3 *Let a plane curve $\mathcal{C}$ and a positive real number d be given. At every nonsingular point A of $\mathcal{C}$, we define a unit normal vector $\vec{N}_A$. The offset of $\mathcal{C}$ at distance d is the geometric locus of points M such that $\vec{AM} = \pm d \cdot \vec{N}_A$.*

Offsets of a plane curve $\mathcal{C}$ are sometimes considered as a special case of envelopes, namely of 1-parameter families of circles of constant radius centered on the curve $\mathcal{C}$. However, they are not identical: offsets are proper subsets of the corresponding envelopes (see [18] for explicit examples and [10] for the different definitions).

2.3 Singularities

The topology of offsets and envelopes is generally much more complicated than that of the progenitor; see [2]. For sufficiently small distances, offsets of an ellipse are smooth, but for larger distances the offsets may have cusps and points of self-intersection (also called *crunodes*); see [23]. Studying these singularities — their detection, classification, and accurate visualization — is a central theme of the present paper.

3 Methodological and Cognitive Framework

3.1 Registers of representation

Duval [25, 26] notes that mathematical objects cannot be grasped with hands, in contrast to what happens in different scientific domains, physics, chemistry, etc. In mathematics, the objects under study are abstract and can be apprehended only via representations. In this work, both parametric representations and implicit equations are used (we may consider them as subregisters of algebraic representation); switching when possible from parametric to polynomial representation. Another classical register is graphical; we noted that even in this register, accuracy of the representation is dependent on the specific kind of algebraic representation. In the computations, we generally obtain a parametric representation of the curve under study. Graphically, the curve is viewed as the union of several components, for which subsequent algebraic manipulations yield the same polynomial equation (see [18, 23]). This is not surprising, as implicitization provides the Zariski closure of the component, and all the components have the same closure. For this switch, we relied heavily on packages for Gröbner bases and Elimination [12]; sometimes resultants are more efficient.

Figure 1 shows the important changes of registers of representation for curves, needed for an efficient study (we use Duval’s terminology).

A 3D printed version makes the object more concrete, with pros and cons. Here a plane curve becomes a 3D object, and then can be manipulated and observed. Note that the examples in this paper are plane curves, but in Figure 1, embedding in 3D space is mentioned. The reason is as follows: a relatively new register of representation for geometrical objects is 3D printing. As a curve (no matter whether it is a plane curve or a space curve) has no width, it has to be thickened in order to be 3D printed, i.e., transformed into a canal surface. This transforms a

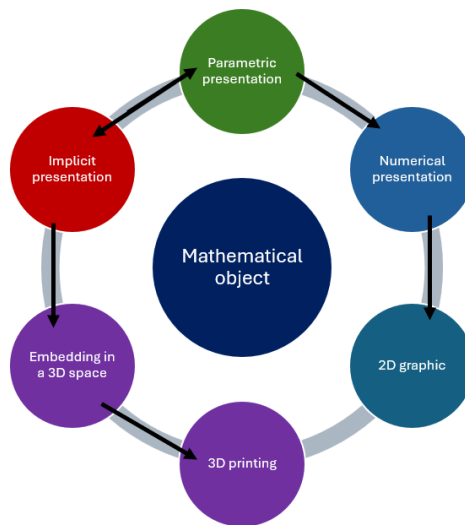


Figure 1: Registers of representation of a plane curve

plane curve into a 3D object. Figure 2 shows two examples: part of a dragon curve³ (important when learning fractals) and a lemniscate⁴.



Figure 2: 3D printed versions of plane curves

3.2 Communication

If in the past, communication was intended between humans, the technology-rich environments available nowadays imply and require new kinds of communication: human-and-machine and machine-and-machine. Already in 2003, Roanes-Lozano et al. [34] wished to have automated communication between kinds of software. Since then, a few advances have been performed, but more developments will be welcome. We still have to copy and paste by hand, i.e., the human is a middleman in the machine-and-machine communication. A student of the first author is currently working to develop an automatic communication.

³Thanks to Zoltan Kovács.

⁴Thanks to Eva Ulbrich.

3.3 The efficiency of new automated commands

The commands relevant to the present paper can be mostly dispatched into two categories: plotting and computing.

- The plotting abilities are often problematic in a neighborhood of a singular point. This is an issue where GeoGebra Discovery's automated command **Plot2D** is useful: not only does it complete the graph by emulating some continuity, but it emphasizes the singular points. Generally, emphasis is placed on cusps and crunodes, but this command also shows elbows, which are much less intuitive. Figure 5 illustrates this.
- As mentioned in Section 1, some commands are numerical, others are symbolic. GeoGebra's **Locus** is numerical, and **LocusEquation** is symbolic. We suggest to read the page <https://geogebra.github.io/docs/manual/en/commands/LocusEquation/>. The validity of the output of a numerical command has often to be asked.

In Subsections 5.1 and 5.2, parametric representations are implicitized using Maple's *PolynomialIdeals* package. This requires a step-by-step work, providing insight into the involved mathematical methods. As noted in Section 1, Maple's package *algcurves* has an **implicitize** command. Its usage can be viewed as a black box usage, but not totally. The command requires a mention of the degree of the desired polynomial. If the "guess" is too low, no output is provided. If it is too large, computation is time consuming, and finally provides a list of polynomials. Paying attention to them, it can be seen that all of them are multiples of a single one, of minimal degree. For example, parametrization $x = \cos t, y = \sin t$ with required degree 2 yields the sole polynomial $x^2 + y^2 - 1$ as expected, but if degree 4 is entered, the following list is obtained:

$$x^2 + y^2 - 1, x^2y + y^3 - y, x^3 + xy^2 - x, x^2y^2 + y^4 - y^2, x^3y + xy^3 - yx, x^4 - y^4 + 2y^2 - 1.$$

It is the educator's role to adopt a way of working, according to the students' background; see [22, 13].

To determine singularities and classify them, a parametrization of the curve is crucial. This is a non-trivial issue, which can be treated using Maple's **parametrization** command, from the *algcurves* package. For a general treatment of the mathematical issues involved here, see [35].

4 Envelopes

In this section, we consider the special case of families of circles of constant radius centered on a given curve, called the progenitor. Then methods can be applied with more complicated examples of 1-parameter families of plane curves, in which cases computations may be much heavier.

We work mostly, but not only, according to the analytic definition of an envelope (Definition 2 in Section 2). Bruce and Giblin [10, Chap. 5] give 4 non-equivalent definitions; here we refer to two of them, with the names given by Kock [27], as recalled in Section 2. It can be shown that an envelope according to Definition 2 is a subset of an envelope according to Definition 1. For example, see [10, 24].

4.1 The progenitor is an ellipse

Take the ellipse $\mathcal{E}$ whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$. It can also be described by a trigonometric parametrization

$$\begin{cases} x(t) = a \cos t \\ y(t) = b \sin t \end{cases} \quad (2)$$

which can be transformed into a rational parametrization, as in [18, 16], using the substitution

$$\begin{cases} \cos t = \frac{1-u^2}{1+u^2} \\ \sin t = \frac{2u}{1+u^2} \end{cases}$$

We consider the family of circles centered on $\mathcal{E}$ with radius r . These circles $\mathcal{C}_t$ have equation $F(x, y, t) = 0$ where $F(x, y, t) = (x - a \cos t)^2 + (y - b \sin t)^2 - r^2$. Figure 3 shows screenshots from the exploration of this family with GeoGebra Discovery, when $r = 1, 2, 3$. The command **Envelope** (**<Path>**, **<Point>**) provides both a plot and a polynomial equation. This equation is of degree 8. Using the **evala(AFactor)** command implemented in Maple, it is shown that this polynomial is irreducible. More details on the computations are to be found in [20, 21]; in particular it is explained there why to use this Maple command and not to factor the polynomial with the CAS embedded in GeoGebra. Since the polynomial is irreducible, the two components which appear in the envelope cannot be distinguished by algebraic means.

Visual analysis of the three cases in Figure 3 leads to the following observations, which have to be assessed later by algebraic computations.

1. For $r = 1$, both components seem to be smooth.
2. For $r = 2$, it seems that the external component is smooth and that the internal has two singular points. Zooming in on the GeoGebra screen helps but is not a proof. To see or not to see, that is the question.
3. For $r = 3$, the topology of the two components is even more different. The external component seems to be smooth, but the internal component is not entirely plotted. Zooming may improve what is seen in a neighborhood of the origin, but leaves open issues.

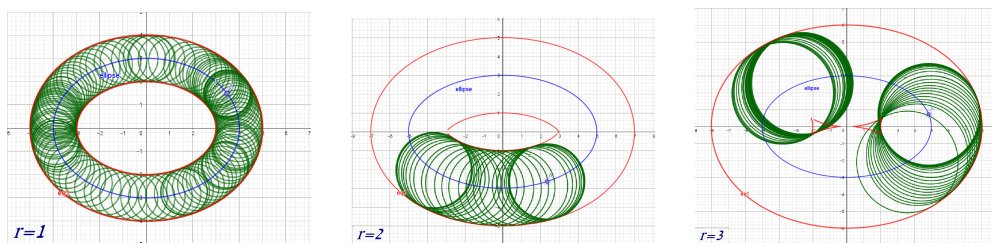


Figure 3: Envelopes of families of circles of constant radius centered on an ellipse

The third case, namely $r = 3$, reveals gaps around the origin and at four other places, where it can be guessed that a cusp should appear. Instead, the plot leaves blank gaps. This situation

suggests that we should use the **Plot2D** command, available in GeoGebra Discovery, but not in the regular GeoGebra.

The output of the **Plot2D** command for $r = 2$ and $r = 3$ is displayed in Figure 4: note that the curve is complete and that the singular points are highlighted. If needed, zooming can be useful.

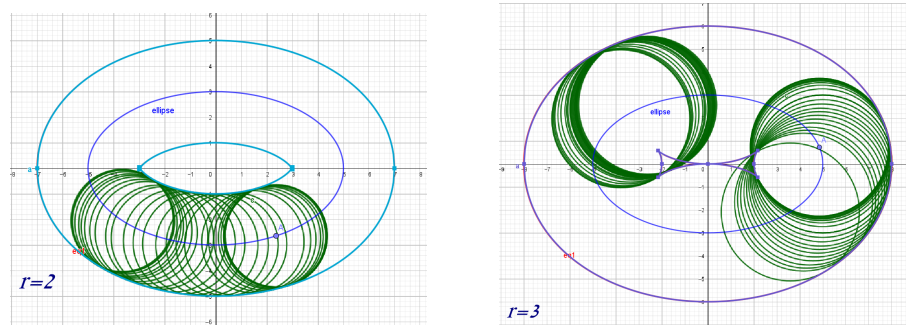


Figure 4: Output with Plot2D

- For $r = 2$, the internal component has singularities at coordinates $(\pm 3, 0)$. The output also shows singular points at the other points of intersection with the x -axis; these are called elbows (see the table classifying singular points in [23]) and attract less interest.
- The improvements obtained with **Plot2D** in the case where $r = 3$ are noticeable and fully fill the gaps in the plot.

Note that in every case, **Plot2D** enhances the singular points. A small pitfall is that this enhancement may hide the fact that there are multiple singular points. This occurs for some values of the radius in our examples, where a small curved triangle appears, whose vertices are two cusps and a crunode.

4.2 The progenitor is a deltoid

This issue has been studied partly in [18]; we mention it in particular because of what will be shown in Subsection 5.2. A GeoGebra applet is available at <https://www.geogebra.org/m/rx2vm2sc>.

A deltoid $\mathcal{D}$ is defined by the following parametric representation

$$\begin{cases} x(t) = a(2 \cos t + \cos 2t) \\ y(t) = a(2 \sin t - \sin 2t) \end{cases} \quad (3)$$

Without loss of generality, we work with $a = 2$ (for aesthetic reasons) and consider the family of circles centered on $\mathcal{D}$ with radius 1. The output is displayed in Figure 5.

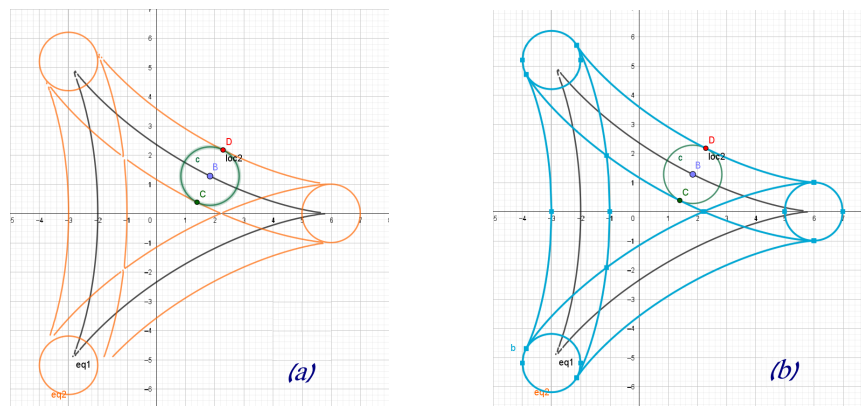


Figure 5: The envelope of unit circles centered on a deltoid

The GeoGebra Discovery session provides the plot (a) and a polynomial equation of degree 14, namely $F(x, y) = 0$, where

$$\begin{aligned}
 F(x, y) = & x^{14} - 32x^{13} + 7x^{12}y^2 + 396x^{12} - 64x^{11}y^2 - 2496x^{11} + 21x^{10}y^4 + 72x^{10}y^2 \\
 & + 14790x^{10} + 160x^9y^4 - 2496x^9y^2 - 140172x^9 + 35x^8y^6 + 564x^8y^4 - 43554x^8y^2 \\
 & + 866060x^8 + 640x^7y^6 + 14976x^7y^4 + 995328x^7y^2 - 2192044x^7 + 35x^6y^8 + 5360x^6y^6 \\
 & - 8772x^6y^4 - 4691920x^6y^2 + 7473633x^6 + 800x^5y^8 + 34944x^5y^6 - 2808504x^5y^4 \\
 & + 2192044x^5y^2 - 68503428x^5 + 21x^4y^{10} + 7476x^4y^8 + 174012x^4y^6 + 2477640x^4y^4 \\
 & + 10557027x^4y^2 + 103434084x^4 + 448x^3y^{10} + 27456x^3y^8 + 3222624x^3y^6 + 10960220x^3y^4 \\
 & + 137006856x^3y^2 + 924501276x^3 + 7x^2y^{12} + 3144x^2y^{10} + 126174x^2y^8 + 7995440x^2y^6 \\
 & + 30330147x^2y^4 + 206868168x^2y^2 - 2633760360x^2 + 96xy^{12} + 7488xy^{10} + 88740xy^8 \\
 & + 6576132xy^6 + 205510284xy^4 - 2773503828xy^2 + y^{14} + 140y^{12} + 1734y^{10} - 40180y^8 \\
 & + 6155425y^6 + 103434084y^4 - 2633760360y^2 + 3375634500.
 \end{aligned} \tag{4}$$

As previously, it can be proven that this polynomial is irreducible. The plot in Figure 5(a) has the following interesting features:

- (i) It is incomplete, i.e. there are probable singularities. This is not a surprise, as the progenitor itself has singularities.
- (ii) There are 3 circles centered at the cusps of the deltoid which contribute arcs to the envelope (actually semi-circles, which has to be proven algebraically). These circles are not part of the curve defined by the polynomial $F(x, y)$. Actually, the obtained envelope corresponds to another definition of envelopes given by [10]; it fits what is needed for safety zones around industrial robots or entertainment parks (see [18]).
- (iii) Note that the 3 circles centered at the cusps of the deltoid do not belong entirely to the envelope, only the semicircles pointing outwards.
- (iv) The "holes" in the plot are larger than in the previous example. They are filled (and the singular points are emphasized) when using **Plot2D**; see Figure 5(b).

5 Offsets

Recall from Section 2 that the offset of a curve $\mathcal{C}$ at distance d is the geometric locus of points at distance d along the unit normal vectors at every nonsingular point of $\mathcal{C}$ (Definition 3). Two components are determined: an "external" (sometimes called *outer offset*) and an "internal" (called *inner offset*). For large enough offset distance, these names may be confusing and roles can be exchanged (see [23]). The topologies of the two components can be different.

5.1 The progenitor is an ellipse

Figure 6 shows offsets of an ellipse whose equation is $\frac{x^2}{25} + \frac{y^2}{9} = 1$, with distances (a) $r = 1$ and (b) $r = 3.3$. The figures have been obtained using GeoGebra Discovery; an applet is available at <https://www.geogebra.org/m/dahfxtdr>. The offsets are defined as geometric loci, therefore we used the **Locus**(**<Point>**,**<Point>**) command. In order to illustrate the dynamic aspect of the work, some of the vectors appear (we used the *Trace On* feature). Two **Locus** commands

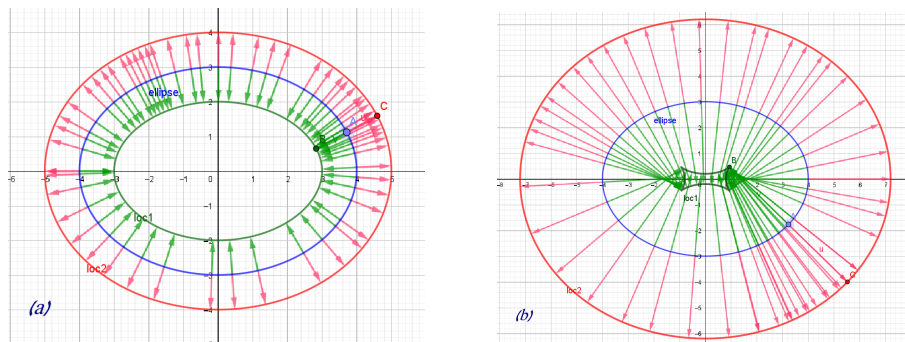


Figure 6: The two offset components can be distinguished

are needed, one for each offset (internal or external). Unfortunately, no symbolic equation is provided. Actually the **Locus** command works numerically. Algebraic computations are required to obtain a parametric representation for each component. A symbolic equation can be derived using Maple, either with its *PolynomialIdeals* package and Elimination or with the **implicitize** command⁵. Both parameterizations yield the same symbolic equation, as it determines the closure of the variety in Zariski topology.

Moreover, even when the progenitor is smooth, the offset can have singularities. This is already the case for offsets of an ellipse, for large enough offset distance. In this case, the regular **Plot** command may not provide an accurate plot, and **Plot2D** provides it with emphasis on singular points. This is the way Figure 6(b) has been obtained.

⁵In [24], the computations are performed by hand. An automated way is used in numerous works, many of them showing advances in implementing automated methods, such as [9, ?, 14, 16, 20, 21, 17].

5.2 The progenitor is a deltoid

We consider a deltoid $\mathcal{D}$ given by the following parametrization:

$$\begin{cases} x(t) = a(2 \cos t + \cos 2t) \\ y(t) = a(2 \sin t - \sin 2t) \end{cases} \quad (5)$$

where a is a positive real (here too, in particular for the figures, we work with $a = 2$). The deltoid can also be defined by a symbolic equation

$$(x^2 + y^2)^2 - 8ax(x^2 - 3y^2) + 18a^2(x^2 + y^2) - 27a^4 = 0. \quad (6)$$

The parametrization (5) provides an accurate plot. Several ways exist to define the curve in GeoGebra; we used the command **Curve** which enables to attach a point A to the curve, then to plot the tangent and the normal to the curve at A , and finally the points of intersection B and C with a circle whose radius is equal to the offset distance. As the construction is totally geometric, the **Locus** command works and provides the 2 components of the offset. Note the following:

- Each component is the disjoint union of arcs, and only the union of all of them is a continuous curve.
- The outer offset and inner offset connect at isolated points, which are cusps of the union, and the role (outer vs. inner) seems to switch at these points.

This is displayed in Figure 7, for a distance equal to 1. Here too, no symbolic equation is obtained for the offset.

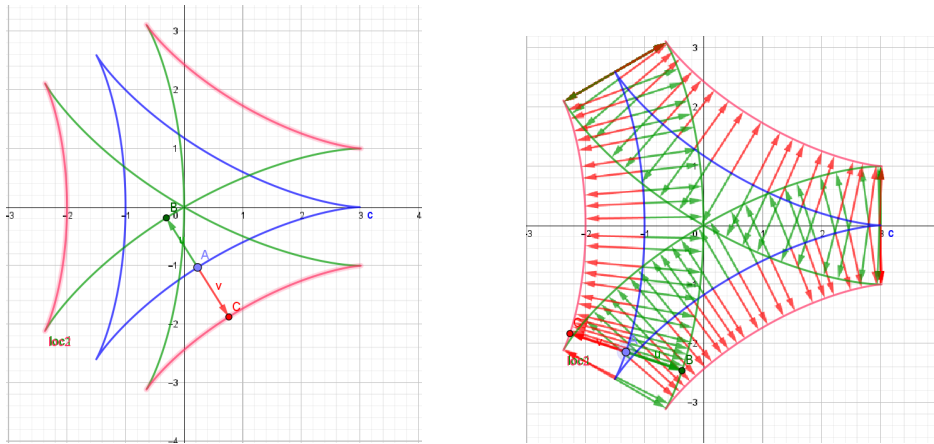


Figure 7: The different arcs of the offsets of a deltoid

From Equation (5), parametric representations can be derived. A tangent vector $\vec{V}(t)$ at A has coordinates

$$\begin{cases} x'(t) = a(-2 \sin 2t - 2 \sin t) \\ y'(t) = a(2 \cos 2t - 2 \cos t) \end{cases} \quad (7)$$

and a normal vector $\vec{N}(t) = (y'(t), -x'(t))$ has coordinates

$$\begin{cases} x_N(t) = a(2 \cos 2t - 2 \cos t) \\ y_N(t) = a(2 \sin 2t + 2 \sin t) \end{cases} \quad (8)$$

A unit normal vector is now $\vec{n}(t) = \frac{1}{\|\vec{N}(t)\|} \vec{N}(t)$. This enables to compute the coordinates of the points B and C described above. This gives actually parametric representations of the loci of B and C when A runs over the deltoid.

The final step consists in transforming the parametric representation into two polynomials, which generate an ideal. By Elimination, a symbolic polynomial equation is derived for the offset. Actually, the parametric representations of both components yield the same symbolic equation, which shows once again that the two components cannot be distinguished by pure algebraic means.

Remark 4 *An ellipse is a smooth plane curve, but a deltoid is not: it has 3 cusps. At these points, a tangent vector and a normal vector are not defined. This is not evident when looking at Figure 7. The existence of singular points on the progenitor has important consequences for the topology of the offset. In [19], an example is given of a very simple smooth progenitor and offsets with multiple cusps and self-intersections.*

6 Experimenting with AI

Every time a new technology emerges, the world of education analyzes it and checks its features and properties to decide if and how to adopt it. Generative AI, particularly in its freely accessible versions, has rapidly gained popularity among students, often independently of institutional policy. In [4], M. Artigue discusses this general issue regarding CAS, but her remarks apply equally to AI. Important studies have already been published; we refer in particular to [7], where affordances of ChatGPT and GeoGebra are analyzed and compared for geometry. Despite the numerous advances in the field, including new algorithms such as the Gröbner Cover [31], the general issue raised in [33, 5] about the decision on "the truth of geometric statements" is still accurate, and comparison of the outcomes of different kinds of software provide an important contribution. Another important issue, related to the previous one, in every study of envelopes and geometric loci with automated methods, is the appearance of irrelevant components. This has already been analyzed in [6], where the authors call them "spurious and genuine parts" of the output. In general, the decision whether a component is relevant or not is made using a mix of algebraic manipulations (often polynomial factorization, as in [20]) and interactive mouse driven work with a DGS.

Therefore, we checked systems used by our students, analyzing their output for the geometric questions addressed in this paper. While some students use Copilot, most of them use ChatGPT⁶ (and some of our colleagues require them to use it for their homework). This is reflected in the amount of work which follows. Our aim is not to benchmark generative AI against CAS/DGS on equal footing, but to document typical failure modes and also the kinds of intermediate outputs that may still be pedagogically useful when explicitly verified. More

⁶OpenAI. (2025). ChatGPT (GPT-5.2) [Large language model]. Online: <https://chat.openai.com>

recently, JCT students use Google Gemini, as the institution made it available for all (the students' technological literacy and skills is also a consequence of the institutional policy; see [4]).

In all the subsections below, human comments and annotations are displayed in *italics*; the bots' outputs appear in the surrounding roman text.

6.1 A first recent experiment with GPT-5.2

More recently, we asked GPT-5.2 for the envelope of a family of circles with constant radius centered on a canonical ellipse. Here is the bot's answer (only the display format has been slightly modified); the titles have been given by the bot, our remarks are in *italics*.

Problem

Let C be the ellipse defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Consider the family of circles of constant radius R , each centered at a point of the ellipse C . Determine the envelope of this family of circles.

The chatbot rephrased the question, this is OK.

Geometric Principle

The envelope of a family of circles with fixed radius r , whose centers move along a smooth curve C , is the *parallel curve* (or *offset curve*) to C at distance r . This curve is obtained by moving a distance r along the normal direction to C at each point.

Thus, the envelope consists of the loci of points at distance r from the ellipse, measured along its normals.

Offsets are often incorrectly described as envelopes; here the bot makes the reverse incorrect shift. Comparing Figures 3 and 6 on the one hand, and Figures 5 and 7 on the other hand, shows that identifying offsets with envelopes of circles with constant radius is sometimes correct but misleading when trying to make a general claim.

Parametrization of the Ellipse

A standard parametrization of the ellipse is
$$\begin{cases} x(t) = a \cos t \\ y(t) = b \sin t \end{cases}, \quad t \in [0, 2\pi).$$

This is correct, of course.

Normal Vector

The ellipse is given implicitly by $F(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$. At each point, the gradient $\nabla F(x, y) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}\right)$ is normal to the ellipse. Substituting the parametrization yields a normal vector proportional to $\left(\frac{\cos t}{a}, \frac{\sin t}{b}\right)$.

The corresponding unit normal vector is
$$\mathbf{n}(t) = \frac{\left(\frac{\cos t}{a}, \frac{\sin t}{b}\right)}{\sqrt{\frac{\cos^2 t}{a^2} + \frac{\sin^2 t}{b^2}}}.$$

This is a good way to compute normals to the curve. It can be performed using a CAS.

Envelope (Parallel Curves)

The envelope consists of two curves, corresponding to the outward and inward normal directions. They are given parametrically by

$$\begin{aligned} x_{\pm}(t) &= a \cos t \pm \frac{r \frac{\cos t}{a}}{\sqrt{\frac{\cos^2 t}{a^2} + \frac{\sin^2 t}{b^2}}}, \\ y_{\pm}(t) &= b \sin t \pm \frac{r \frac{\sin t}{b}}{\sqrt{\frac{\cos^2 t}{a^2} + \frac{\sin^2 t}{b^2}}}. \end{aligned}$$

The method and the answer are correct in this specific case, but the explanation is related to a "wrong" topic, as mentioned above. An unaware user may follow the same process and obtain an offset, which will be a strict subset of the desired envelope. This will occur as soon as the progenitor has singular points.

Remarks

- These curves are known as the *parallel curves* or *offset curves* of the ellipse.
- For sufficiently small r , both curves are smooth.
- When r exceeds the minimum radius of curvature of the ellipse, the inner offset curve develops cusps and self-intersections.
- In general, the envelope does not admit a simple algebraic equation and is best described parametrically.

Parallel curves is the name coined by Leibniz for offsets. The third remark deserves proof and explanation. We do not address here this issue of the limiting value of the radius, and the bot has not been asked about it. Moreover, "to develop" is not a suitable term. The last remark is correct: the solution of the system of equations provides a parametric representation of the curve. According to the question under study, it is enough, or implicitization is needed. The bot does not address this issue.

Conclusion

The envelope of the family of circles of radius r centered on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the union of its inner and outer parallel curves at distance r , given by the parametric equations above.

This does not help much. Moreover, we should emphasize that the ellipse can be represented by the given equation, but it is not the ellipse itself. Duval [26] explains the difference between the mathematical objects and their representations.

6.2 A second recent experiment

Problem

Let C be a deltoid. Consider the family of circles of fixed radius $r > 0$ whose centers move along C . Determine the envelope of this family of circles, describe it explicitly, and analyze its singularities.

Geometric Principle

For any smooth plane curve C , the envelope of circles of radius r whose centers lie on C is the *parallel curve* (or *offset curve*) at distance r . It is obtained by moving a distance r along the normal direction to C .

Thus, the envelope consists of two curves:

- an *outer parallel curve*,
- an *inner parallel curve*.

The approach is different from the previous case, which is not surprising. The bot works on a statistical basis, and it seems that more papers in the world relate to ellipses than to deltoids. The description of outer and inner parallel curves is "new"; it has not been given for the ellipse, despite its generality.

Parametrization of the Deltoid

A standard deltoid (hypocycloid with ratio 3:1) is parameterized by

$$\begin{cases} x(t) = 2a \cos t + a \cos 2t, \\ y(t) = 2a \sin t - a \sin 2t, \end{cases} \quad t \in [0, 2\pi).$$

This curve has three cusps and threefold rotational symmetry.

The mention of a hypocycloid is not necessary. Its positive aspect is to incite the user to learn about this family of curves, but it is not the present concern.

Tangent and Normal

Differentiate:

$$\begin{aligned} x'(t) &= -2a \sin t - 2a \sin 2t, \\ y'(t) &= 2a \cos t - 2a \cos 2t. \end{aligned}$$

A normal vector is obtained by rotating the tangent by $-\pi/2$:

$$\mathbf{N}(t) = (y'(t), -x'(t)) = (2a(\cos t - \cos 2t), 2a(\sin t + \sin 2t)).$$

Its norm is

$$\|\mathbf{N}(t)\| = 2a\sqrt{(\cos t - \cos 2t)^2 + (\sin t + \sin 2t)^2}.$$

Hence a unit normal vector is

$$\mathbf{n}(t) = \frac{(\cos t - \cos 2t, \sin t + \sin 2t)}{\sqrt{(\cos t - \cos 2t)^2 + (\sin t + \sin 2t)^2}}.$$

Writing "a unit normal vector" is more correct.

Envelope (Parallel Curves)

The envelope consists of two offset curves at distance r :

$$\begin{aligned} x_{\pm}(t) &= 2a \cos t + a \cos 2t \pm \frac{r(\cos t - \cos 2t)}{\sqrt{(\cos t - \cos 2t)^2 + (\sin t + \sin 2t)^2}}, \\ y_{\pm}(t) &= 2a \sin t - a \sin 2t \pm \frac{r(\sin t + \sin 2t)}{\sqrt{(\cos t - \cos 2t)^2 + (\sin t + \sin 2t)^2}}. \end{aligned}$$

These are respectively the outer (+) and inner (−) envelopes. *Once again, the same confusion appears in the title, which is not surprising.*

Curvature of the Deltoid

For a deltoid, the curvature is not constant and vanishes nowhere except at cusps. The radius of curvature is given by

$$\rho(t) = \frac{(x'(t)^2 + y'(t)^2)^{3/2}}{|x'(t)y''(t) - y'(t)x''(t)|}.$$

A direct computation shows that: $\rho_{\min} = \frac{a}{3}$, attained midway between the cusps.

No reason is given for including this part about curvature. Actually, curvature of the progenitor is a good mathematical tool to discover cusps. It has been used in [23]. The unaware user does not receive here either an application of curvature or something related to the curvature of the offsets. Note that the notation for curvature is not standard: ρ instead of κ , and midway is not a mathematical notion.

Singularities of the Envelope

- The original deltoid has cusps where the tangent vanishes.
- The **inner parallel curve** develops cusps when $r = \rho(t)$. Thus, for $r \geq \frac{a}{3}$, the inner envelope acquires new cusps and self-intersections.
- For sufficiently large r , even the outer parallel curve develops cusps.

At a cusp no tangent exists; it does not vanish. The bot uses the name tangent in an imprecise (even improper) way, and a curve neither "develops" cusps nor "acquires" cusps. In the case of an ellipse as a progenitor, the chatbot did not mention singularities; this inconsistency illustrates the instability of non-verified explanations. Moreover, the issue is mentioned, but no computation is shown.

Comparison with the Ellipse Case

	Ellipse	Deltoid
Original curve	Smooth	Has cusps
Parallel curves	Smooth for small r	Singular even for small r
Critical radius	Min. radius of curvature	$a/3$
Envelope type	Offset oval	Offset hypocycloid

This could have been a good idea. But the table is incomplete. For example, in the case of the ellipse, one of the offsets (the one which is generally called the outer offset) is smooth, the other one maybe not (for a large radius). The mention of "critical radius" is interesting, but the notion should have been defined; it is generally called the "limiting case".

Conclusion

The envelope of the family of circles of radius r centered on a deltoid is the union of its inner and outer parallel curves at distance r , given parametrically above. Unlike the ellipse, the deltoid's envelope exhibits cusps and self-intersections for comparatively small values of r , reflecting the intrinsic singular nature of the original curve.

The conclusion here is almost nonsense, but interesting. It essentially describes offsets rather than envelopes, as shown in [18].

6.3 A more advanced version of the question with the deltoid

The chatbot proposed some improvements, such as to provide *tikz* pictures (for technical reasons, Figure 8 is a screenshot instead of the original) and analytic computations. Here is the output (it provided a full LaTeX file; we modified the numbering and the style of (sub-sub)sections, and decreased the size of the picture).

Statement of the Problem

Let C be a deltoid. For each point $M \in C$, consider the circle of fixed radius $r > 0$ centered at M . We determine the envelope of this family of circles.

Parametrization of the Deltoid

A standard deltoid (hypocycloid with ratio 3 : 1) is parameterized by

$$\begin{cases} x(t) = 2a \cos t + a \cos 2t \\ y(t) = 2a \sin t - a \sin 2t \end{cases}, \quad t \in [0, 2\pi).$$

The curve has three cusps and a threefold rotational symmetry.

Family of Circles

For each parameter value t , the corresponding circle has equation

$$F(x, y, t) = (x - x(t))^2 + (y - y(t))^2 - r^2 = 0.$$

This defines a one-parameter family of curves.

Until now, this is correct.

Analytic Determination of the Envelope

By definition, the envelope is the locus of points (x, y) for which
$$\begin{cases} F(x, y, t) = 0 \\ \frac{\partial F}{\partial t}(x, y, t) = 0 \end{cases}.$$

This is the analytic definition of an envelope (vs. Def. 2); it appears as one of the non-equivalent definitions in [10, 27]. It is not the general definition of an offset, which was the notion posed in the question.

Compute $\frac{\partial F}{\partial t} = -2(x - x(t))x'(t) - 2(y - y(t))y'(t)$.

Thus the envelope satisfies $(x - x(t))x'(t) + (y - y(t))y'(t) = 0$, which expresses that the vector from the center to the envelope point is orthogonal to the tangent of the deltoid. Hence envelope points lie on the *normal direction*.

Normal Vector

Differentiate:

$$\begin{aligned} x'(t) &= -2a \sin t - 2a \sin 2t \\ y'(t) &= 2a \cos t - 2a \cos 2t \end{aligned}.$$

A normal vector is $\mathbf{N}(t) = (y'(t), -x'(t)) = (2a(\cos t - \cos 2t), 2a(\sin t + \sin 2t))$.

Its norm is $\|\mathbf{N}(t)\| = 2a\sqrt{(\cos t - \cos 2t)^2 + (\sin t + \sin 2t)^2}$.

Here, the analytic definition of an envelope has been abandoned, and the chatbot is back to offsets.

Envelope (Parallel Curves)

The envelope consists of two parallel curves:

$$\begin{aligned} x_{\pm}(t) &= 2a \cos t + a \cos 2t \pm \frac{r(\cos t - \cos 2t)}{\sqrt{(\cos t - \cos 2t)^2 + (\sin t + \sin 2t)^2}}, \\ y_{\pm}(t) &= 2a \sin t - a \sin 2t \pm \frac{r(\sin t + \sin 2t)}{\sqrt{(\cos t - \cos 2t)^2 + (\sin t + \sin 2t)^2}}. \end{aligned}$$

These are respectively the outer and inner envelopes.

The bot continues with the same confusion. And no, these are parametric representations of inner and outer offsets.

Curvature and Cusps

The radius of curvature of the deltoid is $\rho(t) = \frac{(x'(t)^2 + y'(t)^2)^{3/2}}{|x'(t)y''(t) - y'(t)x''(t)|}$.

A direct computation yields $\rho_{\min} = \frac{a}{3}$.

- If $r < \rho_{\min}$, the outer envelope is smooth.
- If $r = \rho_{\min}$, cusps appear.
- If $r > \rho_{\min}$, the inner envelope self-intersects.

Thus singularities arise much earlier than in the ellipse case.

What is meant by "earlier"? Once again, a transition is performed to polar coordinates. This could be interesting if it was well developed and explained. We do not do it, as this is beyond the scope of this paper. Using curvature to determine cusps is a good method; see [23], where Maple code is also displayed.

Figure

The figure provided by ChatGPT is displayed in Figure 8.

It is nice, but incorrect. Compare with Figure 7.

- *At first glance, the segments at the vertices are irrelevant; what seems to be segments are only plots of normal vectors, which are undefined at the vertices of the deltoid.*
- *This comparison shows another confusion: the outer offset and the inner offset are not what is claimed by the bot.*

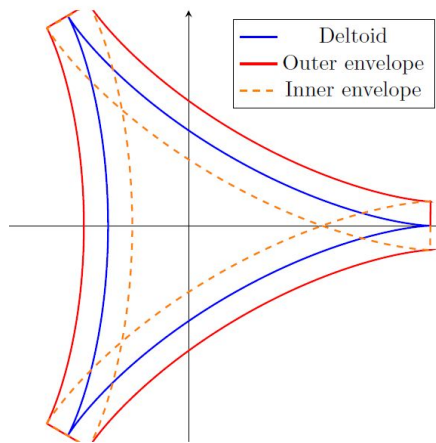


Figure 8: The figure constructed by ChatGPT (GPT-5.2)

Conclusion

The envelope of circles of fixed radius centered on a deltoid is the union of its inner and outer parallel curves. Unlike smooth convex curves (such as ellipses), the deltoid produces envelopes with cusps and self-intersections for relatively small values of r , reflecting its intrinsic singular geometry.

No surprise, it fits the general atmosphere.

6.4 With other AI systems

Initially, we asked a generative AI to analyze the remarks written above about the output by ChatGPT. It confirmed the remarks, without developing an argument. Then we proposed the same question to other systems, such as Claude Sonnet 4.5, Grok 4.1, and Gemini 3 Pro. We summarize some of the pitfalls.

- (i) Each model provided repetitive and imprecise definitions, specifically conflating the term "offset curve" with "parallel curve" in a circular manner that lacks formal descriptive power.
- (ii) There was a persistent failure to distinguish between *offset* and the *envelope*.
- (iii) One model tried to explain what happens with normal vectors at the cusps of the progenitor, but in an unconvincing way.
- (iv) One model erroneously proposed offsets with "variable distance," which is nonsense.
- (v) In trivial cases involving a smooth progenitor such as the ellipse we proposed, the generated figures were technically accurate, though the accompanying theoretical justifications remained superficial or erroneous.
- (vi) While most systems correctly identified the errors in Figure 8, their subsequent attempts at correction were equally flawed. For example, see Figure 9: the half-circles should appear on the envelope, according to Definition 1, but not centered as they appear here. Compare with Figure 5 and the remark which follows there.

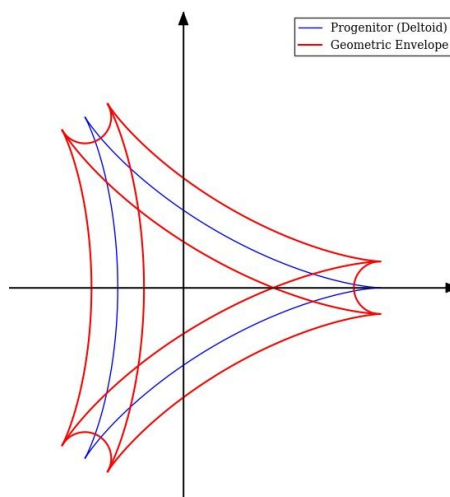


Figure 9: Offset curves of a deltoid, generated via Python code by Claude 3.5 Sonnet.

Google Gemini 3 Pro generated the Python code used to create a figure, supposed to be equivalent to Figure 5; it is displayed in Figure 9. Note the half-circles in a strange position.

7 Educational Experience

7.1 Working with a class and with individuals

The activities described in this work have been proposed to in-service mathematics teachers working toward an advanced degree. Their previous CAS and DGS literacy was quite low; generally they knew how to use a hand-held calculator, but not much more. Nevertheless,

during the course they were able to develop both new technological skills and deeper insight into their previous mathematical knowledge.

At the start, the teacher showed motivating examples of the application of technology to explore mathematics, mostly based on classical theorems in geometry, and a few more advanced constructions, including the use of automated commands that were to be presented in detail in the course. The first reaction was: "wow! Will we learn to do that?". Over time, active communication developed within the group. The suggestion to visit websites devoted to plane curves and to try to explore new curves beyond the formal requirements was received positively, and outcomes were presented. Visits to geometry websites and catalogs of plane curves opened new horizons and helped to develop new skills: how to conduct an autonomous exploration, and acquire self-confidence in building activities. The two main outcomes of the course for in-service teachers were new geometric knowledge and the understanding that technological knowledge is an integral part of new mathematical knowledge. This is a good example of Artigue's claim in [4].

7.1.1 The complementary role of AI in mathematical education

Despite the errors documented in Section 6, generative AI can support mathematical education if used carefully [8]. The key is to treat AI as a starting point for exploration, rather than a source of final truth.

First, AI serves as a partner for brainstorming. As noted by Walkington [39], students can use AI to generate initial ideas, candidate parametrizations, or historical context. This accelerates the initial phase of research.

Second, AI acts as a "safe" conversational partner for communication. Students often struggle to articulate their mathematical arguments; they can use the bot to draft explanations or refine their logic without the pressure of peer or instructor judgment. This lowers the barrier to asking for help and supports the development of mathematical communication skills [32].

Third, analyzing AI errors builds critical thinking. Identifying why the output in Figure 8 is wrong requires students to return to formal definitions. This turns verification into an active learning process, consistent with the instrumental approach described in [4].

Teachers must ensure that students always verify AI outputs (e.g. with theorem-based software). In this workflow, AI provides the initial ideas or text, while CAS and DGS can provide the mathematical proof. The specific workflow described here — using the CAS-verified results of Sections 4 and 5 as ground truth against which AI outputs are measured — provides a concrete instantiation of this general principle that goes beyond commonly reported observations about AI in education.

7.2 Generative AI vs mathematical software

The explorations and proofs presented in this paper have been performed in an environment of CAS and DGS. Using at least two different software packages enabled us to control and verify the exactness of results. These kinds of software work according to implemented mathematical theorems [29, 15, 37], whereas generative AI is optimized for natural-language generation and may produce plausible but unverified statements. The latter does not provide any indication on the sources, but as we asked questions for which the results are known, either from previously

published papers by the authors or by other researchers, it was possible to validate (or not) the output.

7.3 The 4 C's of 21st Century Education

Finally, recall the OECD's description of the 4 C's of 21st century education: communication, collaboration, critical thinking and creativity. Communication is discussed in Subsection 3.2. It is strongly connected to collaboration, a topic enhanced in Section 6. The user may feel that there is a true collaboration between the human and the bot, but this is not the situation: this is not a collaboration between two mathematical reasoners. Here, one is a thinking human, the other is a statistical language model. Therefore, the human must remain in control, which is called critical thinking: not accepting the chatbot's answer "as is". For example, after four iterations the bot claims: "what has been done: ... a rigorous analysis of the envelope ...", which has not really been done. Actually, without a genuine work with CAS and DGS, it is impossible to discover that the bot's developments are incorrect. Applying such critical thinking is required from the educator prior to the activities, as at this step the students do not have the needed knowledge. To know how to behave with a bot is part of the new technological knowledge (see [4]). More research is needed, and new software developments are needed as well.

An interesting part of the conversation with the AI is that it may suggest improvements in the output; this is worth trying, and we did it. This begins with the attitude of the educator, when choosing what to include in teaching, and is part of the didactic transposition, described by Chevallard [11, 3]. Here the educator must be aware of the pitfalls of AI's assisted work (critical thinking), and must be creative to develop a suitable didactic transposition, i.e. what in the teacher's knowledge to transpose into knowledge-to-teach (as an example, we pointed out in Subsection 6.3 that the notion of hypocycloid is superfluous here). Then the educator must help students to develop their own critical thinking, without which creativity may lead to wrong directions. In certain cases, the user has access to the source code, but with AI, no access is available to the sources. Moreover, students may not have a sufficient background to evaluate the quality of the AI's answers, and the teacher's guidance is essential. This aspect is currently under scrutiny and discussion with colleagues and a couple of undergraduate students acting as research assistants.

8 Conclusions

This paper has presented a synergistic approach to analyzing singularities of curve offsets and envelopes, combining Computer Algebra Systems, Dynamic Geometry Software, and generative AI. Three main contributions emerge from this work.

First, we demonstrated on two non-trivial examples — the ellipse and the deltoid as progenitors — that GeoGebra Discovery's **Plot2D** command effectively resolves the plotting failures that occur near singular points of envelopes and offsets, where the standard **Plot** command leaves blank gaps. This capability is essential for accurate visual exploration of curve topology.

Second, we provided a systematic documentation of how several generative AI systems (ChatGPT GPT-5.2, Claude Sonnet, Grok, and Gemini) handle questions about envelopes and offsets. A persistent failure across all tested models is the conflation of offsets with envelopes

— a subtle but mathematically significant distinction. The generated figures and analytic developments contained errors that can only be detected through verification with theorem-based software.

Third, we described and deployed a verification workflow in which CAS and DGS serve as ground truth against which AI outputs are checked. This workflow was embedded in educational activities with in-service mathematics teachers and undergraduate research assistants, yielding qualitative evidence that it fosters critical thinking and deeper engagement with formal definitions.

Looking ahead, further development is needed in automated communication between CAS and DGS to reduce the manual copy-and-paste steps that remain necessary. The integration of AI verification workflows into mathematics coursework deserves further systematic study, particularly regarding the level of mathematical background students need to evaluate AI outputs effectively.

Declarations

No conflict of interest.

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